NOTE ON LOGISTICS EQUATION

If you find any errors or typos in this note, please alert me.

The logistics equation is

$$\frac{dy}{dt} = ky(M - y),$$

where k and M are positive constants that we interpret in the following way. This equation is used to model things like spread of a disease, or spread of information, and then M represents the total number of people and $y(t)$ is the function that measures the number of people infected at time t, or the number of people who know the information at time t, respectively. The differential equation thus says that the rate of growth of the disease (or the spread of the information) is proportional to the product of the number of people infected (or people who know the information) and the number of people not infected (or people who don't know the information), and the proportion is the constant k.

Note 1. This is also example 6 from 8.1.2 in you book. However, I am treating and interpreting the constants a little differently because I am finding this interpretation more enlightening and easier to understand. You are encouraged to read both approaches. The equation and solution are of course the same.

Let's solve this equation in terms of M and k. Supposing that $y(t) \neq 0$ and $y(t) \neq M$ (those would be the cases when no one or everyone is infected), we can divide by $y(M - y)$ to separate the variables to get

$$\int \frac{1}{y(M-y)} dy = \int k dt,$$

where on the oft we have used the substitution $y = y(t); \quad dy = y'(t)dt$.

By partial fraction decomposition, we get

$$\frac{1}{y(M-y)} = \frac{1/M}{y} + \frac{1/M}{M-y}.$$

Thus

$$\frac{1}{M}\Big(\int \frac{1}{y} dy + \int \frac{1}{M-y} dy\Big) = kt + C_0.$$

Thus

$$\frac{1}{M}(\ln|y| - \ln|M - y|) = kt + C_0$$

Note 2. The constants from the partial fraction decomposition are both $\frac{1}{M}$. I am sorry, this is what I messed up in class because I got confused when looking at my own notes about the minus sign in the last line.

Note that I got the minus sign in the last line because the derivative of $M-y$ is -1, and what I had done was a substitution to compute the second integral.

Since $y(t)>0$ and $y(t)<M$, we can drop the absolute value signs. We get

$$\ln \frac{y}{M-y} = Mkt + MC_0,$$

thus

$$\frac{y}{M-y} = Ce^{Mkt},$$

where $C = e^{MC_0}$ can be any positive constant.

Solving for y we get

$$y(t) = \frac{CM}{C + e^{-Mkt}}.$$

(I've skipped some of the steps in the algebra, make sure you can derive this yourself.)

Now suppose we are given an initial value constraint $y(0) = R$, so R represents the number of people infected to begin with. Then

$$R = \frac{CM}{C+1}, \qquad \text{so} \qquad C = \frac{R}{M-R}.$$

Thus

$$y(t) = \frac{MR}{R + (M-R)e^{Mkt}},$$

and as we discussed in class, $y(t) \to M$ as $t \to \infty$, so according to this model the disease (or information) will tend to infect or spread to all people.